ON MULTIPLICITIES FOR SL(n)

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ABSTRACT

We use the strong Artin conjecture for Galois extensions of Heisenberg type to show that a cuspidal automorphic representation of SL(N)/F, for F a number field and $N > 2$, can occur with multiplicity greater than one. We also exhibit two cuspidal L-packets (for $F = Q$ and N prime) which are locally isomorphic for primes p different from N, but which are disjoint at N, i.e. that L-packets are not rigid.

Introduction

A basic problem in the theory of automorphic forms is the determination of the multiplicity with which a given isomorphism class of representation occurs in the space of cusp forms. In this paper we construct families of examples on the groups SL_n, $n \geq 3$, for which the multiplicities are greater than 1.

To describe the results, let F be a number field with adele ring $\mathbb{A}_F$. Let $A_0(\mathrm{SL}_n(\mathbb{A}_F)) \subseteq L^2(\mathrm{SL}_n(F)\backslash\mathrm{SL}_n(\mathbb{A}_F))$ denote the subspace of cusp forms. Then $A_0(\mathrm{SL}_n(\mathbb{A}_F))$ is a fully and discretely decomposable $\mathrm{SL}_n(\mathbb{A}_F)$-module

$$A_0(\mathrm{SL}_n(\mathbb{A}_F)) \xrightarrow{\sim} \widehat{\bigoplus_{\pi}} m(\pi) \cdot \pi$$

with pairwise inequivalent irreducible admissible (unitary) $\mathrm{SL}_n(\mathbb{A}_F)$-modules π and multiplicities $m(\pi) \in \mathbb{Z}$, $0 \leq m(\pi) < \infty$. The first main result of this paper is the following:

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THEOREM A:

(a) *Let n be an integer, $n \geq 3$. If n is odd or if 4 divides n, then for each number field F, there exist infinitely many irreducible cuspidal automorphic representations π on $\mathrm{SL}_n(\mathbb{A}_F)$ with $m(\pi) > 1$.*

(b) *If $n = 2m$ with $m \geq 3$ odd, then there exist infinitely many number fields F for which there is a cuspidal automorphic representation π on $\mathrm{SL}_n(\mathbb{A}_F)$ with $m(\pi) > 1$.*

We now outline the ideas needed to prove this result. Let π be an irreducible constituent of $A_0(\mathrm{SL}_n(\mathbb{A}_F))$. Then π can be extended ([LL]) to a cuspidal automorphic representation $\hat{\pi}$ of $\mathrm{GL}_n(\mathbb{A}_F)$. Let $\mathcal{L}(\pi)$ be the (semisimple) representation of $\mathrm{SL}_n(\mathbb{A}_F)$ obtained by restricting the action of $\mathrm{GL}_n(\mathbb{A}_F)$ on $\hat{\pi}$ to $\mathrm{SL}_n(\mathbb{A}_F)$. The isomorphism class of $\mathcal{L}(\pi)$ is independent of the choice of $\hat{\pi}$ and each constituent of $\mathcal{L}(\pi)$ occurs with multiplicity one (cf. [LL]; the proofs given there for $n = 2$ remain unchanged for general n). It is known that $\mathcal{L}(\pi) \xrightarrow{\sim} \mathcal{L}(\pi')$ if and only if, for each place v of F, $\hat{\pi}'_v \xrightarrow{\sim} \hat{\pi}_v \otimes \psi_v$ with a character ψ_v of $\mathrm{GL}_n(F_v)$.

Now let $L(\pi) \subseteq A_0(\mathrm{SL}_n(\mathbb{A}_F))$ be the quotient of $\mathcal{L}(\pi)$ obtained as the closure of the restriction of the smooth functions in the space of $\mathcal{L}(\pi)$ to $\mathrm{SL}_n(\mathbb{A}_F)$. It is known ([LL]) that $L(\pi) \cap L(\pi') \neq \{0\}$ if and only if there exists a character ψ: $\mathrm{GL}_n(\mathbb{A}_F) \to \mathbb{C}^*$, trivial on $\mathrm{GL}_n(F)$, i.e. an ideal class character of $\mathbb{A}_F^*$, such that $\hat{\pi}' \xrightarrow{\sim} \hat{\pi} \otimes \psi$. In this case, $L(\pi) = L(\pi')$. Furthermore, for a set of representatives for the π's modulo this equivalence relation, the $L(\pi)$'s give a direct sum decomposition of $A_0(\mathrm{SL}_n(\mathbb{A}_F))$. The previous theorem is an easy consequence of the following assertion (cf. Prop. 3.3 below):

For each $n \geq 3$ and each number field F there exist infinitely many pairs π, π' occurring in $A_0(\mathrm{SL}_n(\mathbb{A}_F))$ with

$$\text{(a)} \quad \mathcal{L}(\pi) \xrightarrow{\sim} \mathcal{L}(\pi'),$$
$$\text{(b)} \quad L(\pi) \cap L(\pi') = \{0\}.$$

This claim is simply the SL_n formulation of the assertion:

For each $n \geq 3$ and each number field F there exist infinitely many pairs $\hat{\pi}, \hat{\pi}'$ in $A_0(\mathrm{GL}_n(\mathbb{A}_F))$ such that

(a) *for all places v, there exists a character φ_v: $F_v^* \to \mathbb{C}^*$ such that*

$$\hat{\pi}_v \xrightarrow{\sim} \hat{\pi}'_v \otimes \psi_v,$$

(b) *there does not exist an idele class character* $\psi\colon \mathbb{A}^*_{F/F^*} \to \mathbb{C}^*$ *such that*

$$\hat{\pi} \xrightarrow{\sim} \hat{\pi}' \otimes \psi\,.$$

This latter assertion is proven by constructing certain nilpotent Galois representations (Section 2), invoking the strong Artin conjecture ([AC]), and analysing ramification of the resulting π's (Section 4).

Theorem A would follow at once if we knew the following

CONJECTURE: *For all n and F, if $\mathcal{L}(\pi)$ is isomorphic to $\mathcal{L}(\pi')$, then $L(\pi)$ is isomorphic to $L(\pi')$.*

For $n = 2$, this conjecture follows from the stable trace formula ([LL]) and in this case it is standard to hope that $L(\pi)$ actually equals $L(\pi')$, i.e. all multiplicities are one. Lacking the conjecture, we resort to a trick using complex conjugation ρ. Using the additive characters relative to which the constituents of $\mathcal{L}(\pi)$ have Whittaker models we show that we can make π as above for which $\mathcal{L}(\pi^\rho) \cong \mathcal{L}(\pi)$ but $L(\pi) \cap L(\pi^\rho) = \{0\}$, where ρ denotes complex conjugation. Indeed, we can parametrize a constituent η of $\mathcal{L}(\pi)$ by the family of additive characters ψ relative to which η has a Whittaker model. Then it is enough to show that: 1. for a member τ of $L(\pi)$, τ^ρ has Whittaker model for the same ψ as τ, and 2. $(\hat{\pi})^\rho$ is not a global twist of $\hat{\pi}$, where $\hat{\pi}$ is an extension of π to a cuspidal representation of $\mathrm{GL}_n(\mathbb{A}_F)$.

Let π_v be an irreducible admissible representation of $\mathrm{SL}_n(F_v)$, and define $\mathcal{L}(\pi_v)$ as in the global case.

THEOREM B: *Let p be an odd prime. Then there exist π, $\pi' \subset A_0(\mathrm{SL}_p(\mathbb{A}_Q))$ such that*

$$\mathcal{L}(\pi_v) \xrightarrow{\sim} \mathcal{L}(\pi'_v) \quad (v \neq p),$$

and

$$\mathcal{L}(\pi_p) \text{ is not isomorphic to } \mathcal{L}(\pi'_p)\,.$$

This theorem shows that no "strong multiplicity one", i.e. rigidity, result holds for the collections (L-packets) $\mathcal{L}(\pi)$. Of course, the assertion that strong multiplicity one fails at the level of the representations themselves is implied by the fact that the $L(\pi)$ should not, in general, be irreducible. This latter result (the occurrence of L-indistinguishability) is known for SL_2 and certain other groups. The proof of Theorem B is a variant of that of Theorem A.

The cusp forms we construct are endoscopic for SL_n (i.e. are automorphically induced from representations on $\mathrm{SL}_n(\mathbb{A}_L)$) where L is a cyclic extension. It is natural to ask whether there exists an analogous construction of stable (i.e. non-endoscopic) representations. An example of Borovic ([GW]) suggests, by the method of this paper, that this is so, and hence that $m(\pi) > 1$ more generally.

The whole paper is motivated by our attempt to understand one aspect of the multiplicity conjecture of Arthur, Langlands, and Shelstad ([A]) in the context of [LL] and SL_n. In particular, I wish to thank J.-P. Labesse for discussions on this subject. Also, I thank J.-P. Serre for introducing me to the nilpotent groups H_p which are crucial to our construction.

1. Heisenberg groups

1.1 Let $n \geq 3$ be an integer, and let $e_1, \ldots, e_n$ denote a basis of $\mathbb{C}^n$. Let H_n be the finite group with generators A, B and C and relations:

$$\begin{aligned} A^n &= B^n = C^n = 1, \\ AC &= CA, \\ BC &= CB, \\ AB &= CBA. \end{aligned}$$

Let $\alpha \in \mathbb{Z}$ with $(\alpha, n) = 1$. Define a faithful linear representation ρ_α of H_n by

$$\begin{aligned} \rho_\alpha(A)e_i &= \xi_n^{(i-1)\alpha} e_i \qquad (1 \leq i \leq n), \\ \rho_\alpha(B)e_i &= e_{i+1} \qquad (1 \leq i \leq n-1), \\ \rho_\alpha(B)e_n &= e_1, \\ \rho_\alpha(C)e_i &= \xi_n^\alpha e_i, \end{aligned}$$

where $\xi_n = e^{2\pi i/n}$. Note that for $Z = \langle C \rangle$, if $h \in H_n$, $h \notin Z$, $\mathrm{Tr}(\rho_\alpha(h)) = 0$. Hence $\langle \chi_\alpha, \chi_\alpha \rangle_{H_n} = 1$, where χ_α is the character of ρ_α, and so each ρ_α irreducible. Since Z acts differently in each ρ_α, they are pairwise inequivalent. For $h \in H_n$, let $a = a(h)$ be the least integer such that $h^a \in Z$. Suppose $h^a = C^b$. Then $(\xi_{na}^{-b\alpha} \rho_\alpha(h))^a = 1$. Hence, for any $\beta \in \mathbb{Z}$, $(\beta, n) \neq 1$,

$$\mathrm{Tr}((\xi_{na}^{-b\alpha} \rho_\alpha(h))^m) = \mathrm{Tr}((\xi_{na}^{b\beta} \rho_\beta(h))^m)$$

for all $m \geq 0$. This follows because if a does not divide m, the traces are zero, and if a divides m, the traces are n. Since the numbers $\mathrm{Tr}(X^n)$ for all m determine the conjugacy class in GL_n of a semisimple element, this means $\rho_\alpha(h)$ is conjugate to $\xi_{na}^{b(\alpha-\beta)}\rho_\beta(h)$ in $\mathrm{GL}_n(\mathbb{C})$. In particular, if $\overline{\rho_\alpha(h)}$ denotes the image of $\rho_\alpha(h)$ in $\mathrm{PGL}_n(\mathbb{C})$, the conjugacy class of $\overline{\rho_\alpha(h)}$ is independent of α. Note however that the representations $\bar{\rho}_\alpha$ are all inequivalent in $\mathrm{PGL}_n(\mathbb{C})$. Indeed, if $\bar{g} \in \mathrm{PGL}_n(\mathbb{C})$, the commutator of any lifts to $\mathrm{GL}_n(\mathbb{C})$ of $\bar{g}\bar{\rho}_\alpha(A)\bar{g}^{-1}$ and $\bar{g}\bar{\rho}_\alpha(B)\bar{g}^{-1}$ is ξ_n^α.

1.2 To avoid a complicated treatment, we now restrict our analysis to the cases where $n = 4$ or n is an odd prime. Suppose first that n is an odd prime. Then any proper subgroup $T \subsetneq H_n$ is abelian and either

$$\begin{aligned} & T = \langle C \rangle, \\ \text{or} \quad & T = \langle h \rangle \qquad (h \text{ non-central}), \\ \text{or} \quad & T = \langle h, C \rangle \quad (h \text{ non-central}). \end{aligned}$$

In each case, it is clear that the isomorphism class in $\mathrm{PGL}_n(\mathbb{C})$ of $\bar{\rho}_\alpha|_T$ is independent of α. Furthermore, for each pair α, β there is a character $\chi_{\alpha,\beta}\colon T \to \mathbb{C}^*$ such that $\rho_\alpha|T$ is isomorphic to $\rho_\beta|_T \otimes \chi_{\alpha\beta}$.

Suppose now that $n = 4$. Consider a proper subgroup $T \subsetneq H_4$ of the form $(\mathbb{Z}/2^a\mathbb{Z}) \rtimes (\mathbb{Z}/2^b\mathbb{Z})$ (semidirect product). Then $0 \leq a, b \leq 2$ since every element in H_4 has order 4. If T is non-abelian, we must have $a = 2$. Then the image of T in $H_4/Z \xrightarrow{\sim} \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ must be a proper subgroup which contains an element x of order 4. Otherwise $T = H_4$, or T is commutative, since A^2 commutes with B^2. The image of T must strictly contain the subgroup $\langle x \rangle$, since otherwise T would again be abelian. Hence the image is of the form $\langle x, y \rangle$ with $y^2 = 1$. Choosing preimages $\tilde{x}$ and $\tilde{y}$, we find $[\tilde{x}, \tilde{y}] = C^2$, and hence the center of T is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$. However the center of a non-abelian group $(\mathbb{Z}/4\mathbb{Z}) \rtimes (\mathbb{Z}/2^b\mathbb{Z})$ is either $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ $(b = 1)$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ $(b = 2)$. Hence T is abelian, and its image in H_4/Z is isomorphic to (a) $\{0\}$, (b) $\mathbb{Z}/4\mathbb{Z}$, (c) $\mathbb{Z}/2\mathbb{Z}$, or (d) $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. In cases (a), (b), and (c), we conclude as before that $\rho_1|_T \xrightarrow{\sim} \rho_3|_T \otimes \chi_{13}$ with a character $\chi_{13}\colon T \to \mathbb{C}^*$. In case (d), suppose first $T \cap Z = 1$. Since $\rho_1(\pm A^2) = \rho_3(\pm A^2)$ and $\rho_1(\pm B^2) = \rho_3(\pm B^2)$, and $T = \langle \pm A^2, \pm B^2 \rangle$, $\rho_1|_T = \rho_3|_T$. If $T \cap Z \neq 1$, then, under our hypothesis on T, $T \cap Z = \mathbb{Z}/2\mathbb{Z}$ and since $\chi_1(C^2) = -2$, $\chi_1|_T$ is real. Since $\chi_3 = \bar{\chi}_1$, this shows that $\chi_1|_T$ is equivalent to $\chi_3|_T$.

Note finally that if $T \xrightarrow{\sim} \mathbb{Z}/n^a\mathbb{Z} \rtimes \mathbb{Z}/n^b\mathbb{Z}$ with n odd, then $a, b \leq 1$ and in particular $T \neq H_p$. Similarly, H_4 is not isomorphic to $\mathbb{Z}/2^a\mathbb{Z} \rtimes \mathbb{Z}/2^b\mathbb{Z}$ for any a, b. We summarize this discussion.

PROPOSITION 1: *Let n be an odd prime or $n = 4$. Let T be a subgroup of H_n and suppose that T is isomorphic to a subgroup of $\mathbb{Z}/n\mathbb{Z} \rtimes \mathbb{Z}/n\mathbb{Z}$ (n odd) or $\mathbb{Z}/2^a\mathbb{Z} \rtimes \mathbb{Z}/2^b\mathbb{Z}$ ($n = 4$). Then T is a proper subgroup of H_n and, for each α, $\beta \in \mathbb{Z}$, $(\alpha, n) = 1$, $(\beta, n) = 1$, there exists a character $\chi_{\alpha\beta}\colon T \to \mathbb{C}^*$ such that*

$$\rho_\alpha|_T \xrightarrow{\sim} \rho_\beta|_T \otimes \chi_{\alpha\beta}.$$

2. Galois representations

2.1 PROPOSITION: *Let n be an odd prime or $n = 4$. Let K be a number field and let S be a finite subset of the finite places of K, including those which are above the prime dividing n. Then there exist infinitely many Galois extensions L of K with $\mathrm{Gal}(L/K) = H_n$ such that*

(1) *L is unramified over each place v in S,*

(2) *for each place v of K and each $\alpha, \beta \in (\mathbb{Z}/n)^*$, we have*

$$\rho_\alpha|_{D_v} \xrightarrow{\sim} \rho_\beta|_{D_v} \otimes \chi_{\alpha,\beta}$$

with a character $\chi_{\alpha,\beta}\colon D_v \to \mathbb{C}^*$, where $D_v \subseteq H_n$ is a decomposition group for v,

(3) *$\bar{\rho}_\alpha$ is not isomorphic to $\bar{\rho}_\beta$ unless $\alpha = \beta$.*

2.2 Proof: By Shafarevich's theorem, there exist infinitely many extensions L with $\mathrm{Gal}(L/K) \xrightarrow{\sim} H_n$ which are unramified (even split-completely) at the places in S. For $v \notin S$, the wild inertia subgroup of D_v is trivial, since v is prime to n. Hence L is tamely ramified at v and so D_v is a quotient of $(\mathbb{Z}/(q^f - 1)\mathbb{Z}) \rtimes \mathbb{Z}/m\mathbb{Z}$ where, if F_v denotes a generator of $\mathbb{Z}/m\mathbb{Z}$, $F_v x F_v^{-1} = qx$ ($x \in \mathbb{Z}/(q^f - 1)\mathbb{Z}$). Here q is the number of elements in the residue field of K at v, and $f \geq 1$. Hence, since each element of H_n has order dividing n, D_v is isomorphic to $\mathbb{Z}/n^a\mathbb{Z} \times \mathbb{Z}/n^b\mathbb{Z}$ ($a, b \leq 1$) if n is odd, or $\mathbb{Z}/2^a\mathbb{Z} \times \mathbb{Z}/2^b\mathbb{Z}$ ($a, b \leq 2$) if $n = 4$. The claim now follows at once from Proposition 1 for the finite places. If v is infinite, and n is odd, $D_v = \{1\}$. If $n = 4$, D_v is at most $\mathbb{Z}/2\mathbb{Z}$ and we see easily that $\rho_1|_{D_v} \xrightarrow{\sim} \rho_2|_{D_v}$. This proves the claim. ∎

2.3 Corollary: *Let $N \geq 3$ and write $N = 2^a 4^b \prod_{i=1}^m p_i$ with $a \in \{0,1\}$, $b \geq 0$, and odd primes p_i, not necessarily distinct. Put $m(n) = 2^b \prod_{i=1}^m (p_i - 1)$. Let K be a number field and let S be a finite set of finite places of K which contains those which divide N. Then there exist infinitely many Galois extensions L of K with*

$$\mathrm{Gal}(L/K) \simeq G_N = (\Delta)^a \times H_4^b \times \prod_{i=1}^m H_{p_i}$$

where Δ is a dihedral group $\mathbb{Z}/t\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$, $t > 2$, such that

(a) *L is unramified at the places in S.*

(b) *$\mathrm{Gal}(L/K)$ has $m(n)$ pairwise non-isomorphic irreducible representations*

$$\rho_i \colon G_N \to \mathrm{GL}_N(\mathbb{C}) \qquad (1 \leq i \leq m(n))$$

such that for every place v, and each pair (i,j), there exists a character $\chi_{ij} \colon D_v \to \mathbb{C}^*$ such that

$$\rho_i|_{D_v} \xrightarrow{\sim} \rho_i|_{D_v} \otimes \chi_{ij,v}\,.$$

(c) *$\bar{\rho}_i$ is not isomorphic to $\bar{\rho}_j$ unless $i = j$.*

(d) *D_v is abelian for all v.*

Proof: Let K_1 be a quadratic imaginary extension of Q which is unramified at the primes of $\mathbb{Q}$ under the primes of S, and such that $t > 2$ divides the order of the class group of K_1. Let χ_1 be an ideal class character of order t, viewed as a Galois character. Then $\mathrm{Ind}_{K_1}^{Q}(\chi_1)$ is an irreducible dihedral representation of $\mathrm{Gal}(L_1/\mathbb{Q}) = \Delta$. Note that all decomposition groups are abelian since χ_1 is unramified. Choose K_1 and χ_1 so that $\mathrm{Gal}(L_1K/K) = \Delta$. Next, invoke Proposition 2.1 to construct an extension L_2 with Galois group $H_4^b \times \prod_{i=1}^m H_{p_i}$. Arrange that L_2 is linearly disjoint from L_1 over K. Put $L = L_1L_2$. Then $\mathrm{Gal}(L/K) \simeq G_N$. Let ρ_Δ be an irreducible two dimensional representation of Δ. Let the ρ_i be the $m(n)$ evident tensor products of the representations ρ_α, and e_Δ, if $a = 1$, choosing one ρ_α for each factor H_n. Then all claims follow easily. ∎

2.5 Proposition: *Let n be an odd prime. Let K be a number field which does not contain any non-trivial roots of unity of order n and let S be a subset of the places of K which divide n. Then there exist infinitely many Galois extensions L of K with $\mathrm{Gal}(L/K) \xrightarrow{\sim} H_n$ such that*

(a) $D_v = H_n \qquad (v \in S)$,
(b) $D_v \subsetneq H_n \qquad (v \notin S)$,
(c) $\rho_\alpha|_{D_v} \xrightarrow{\sim} \rho_\beta|_{D_v} \otimes \chi_{\alpha\beta,v} \qquad (v \notin S)$,
(d) $\bar{\rho}_\alpha \xrightarrow{\sim} \bar{\rho}_\beta$ *only if* $\alpha = \beta$.

Proof: Since n is odd and H_n is nilpotent, we can apply the theorem of Neukirch ([N]) to ensure that $D_v \xrightarrow{\sim} H_n$ if $v \in S$, and $D_v \neq H_n$ if $v \notin S$ but v lies over n. Since L is tamely ramified for $(v, n) = 1$, we conclude as before that $D_v \neq H_n$ if $v \notin S$, and the result follows. ∎

3. Automorphic forms

3.1 PROPOSITION: *Let $\rho = \rho_N \otimes \rho_\Delta$ be an irreducible representation of a Galois group $\mathrm{Gal}(L/K) = N \times \Delta$ where N is nilpotent and Δ is dihedral. Then there exists a cuspidal automorphic representation $\pi(\rho)$ of $\mathrm{GL}_m(\mathbb{A}_K)$, $m = \dim(\rho)$, such that for all but finitely many places v of K*

$$L_v(\rho, s) = L_v(\pi(\rho), s).$$

3.2 Proof: If $\Delta = \{1\}$, this is Theorem (7.1) of [AC]. If $\Delta \neq \{1\}$, then $\rho_\Delta \xrightarrow{\sim} \mathrm{Ind}_Q^K(\psi)$ where Q is a quadratic extension of K and ψ is a character of $\mathrm{Gal}(L/Q)$. Hence $\rho \xrightarrow{\sim} \mathrm{Ind}_Q^K((\rho_N)|_Q \otimes \psi)$. Let $\tilde{\psi}$ be the idele class character associated to ψ. Then $\eta = \pi((\rho_N)|_Q) \otimes \tilde{\psi}$ is a cuspidal representation of $\mathrm{GL}_{m/2}(\mathbb{A}_Q)$ for which $\eta \circ \tau$ is not isomorphic to η, where τ denotes the non-trivial automorphism of Q over K. By [AC], there exists a unique cuspidal representation $\pi(\rho)$ of $\mathrm{GL}_m(\mathbb{A}_K)$ such that

$$L_v(\pi(\rho), s) = \prod_{w|v} L_w(\pi((\rho_N)|_Q \otimes \tilde{\psi}, s) = \prod_{w|v} L_w((\rho_N)|_Q \otimes \psi, s) = L_v(\rho, s)$$

for almost all v. This proves the proposition. ∎

Note that such a $\pi(\rho)$ is unique by the strong multiplicity one theorem.

3.3 PROPOSITION: *Let K be a number field and let $N \geq 3$. Then there exist infinitely many disjoint families $\mathcal{F} = \{\pi_j \mid 1 \leq j \leq m(n)\}$ of cuspidal automorphic representations of $\mathrm{GL}_N(\mathbb{A}_K)$ such that*

(a) *for each place v of K, there exists a character $\chi_{ij,v}\colon K_v^* \to \mathbb{C}^*$ such that*

$$\pi_{i,v} \xrightarrow{\sim} \pi_{j,v} \otimes \chi_{ij,v},$$

(b) *there does not exist an idele class character* $\chi_{ij}\colon \mathbb{A}_K^*/K^* \to \mathbb{C}$ *such that*

$$\pi_i \xrightarrow{\sim} \pi_j \otimes \chi_{ij}\,.$$

3.4 Proof: Let $\mathcal{T}$ be a family of representations $\pi_i = \pi(\rho_i)$ with a collection $\{\rho_i | 1 \le i \le m(n)\}$ as Corollary (2.3). At almost all unramified places v of K we have $L_v(\pi(\rho_i), s) = L_v(\rho_i, s) = L_v(\rho_j \otimes \chi_{ij}, s) = L_v(\pi \otimes \chi_{ij}, s)$ where χ_{ij} has been regarded as both a character of D_v and of K_v^* by local class-field theory. To continue we need to relate $\pi(\rho_i)_v$ to $\rho_i|D_v$ at the other places. In our case, the needed result is well-known (cf. [AC], Remark after (7.11)) to the experts, but we do not know a convenient reference. We derive what we need from [AC].

3.5 Lemma: *Let v be a place of K and let χ_v be a character of finite order of K_v^*, identified via local class-field theory with a character of* $\mathrm{Gal}(K_v^{ab}/K_v)$. *Let $N \ge 3$, and let $\rho_i = \rho\colon G_N \to \mathrm{GL}_N(\mathbb{C})$ with a $\rho_i \in \mathcal{F}$. Then*

$$L_v(\rho|_{D_v} \otimes \chi_{v,s}) = L_v(\pi(\rho) \otimes \chi_{v,s})\,.$$

3.6 Proof: Each irreducible factor of the tensor expression for ρ is induced from a character of a normal subgroup $T \subseteq \Delta$ or H_n with Δ/T or H_n/T cyclic. Evidently, ρ itself is induced from a character ψ of the product T' of these subgroups and H_N/T' is just a product of quotients of the form Δ/T and H_N/T. Let F be a solvable finite extension of K such that $\rho|_F$ is irreducible and unramified. Then $\pi(\rho|_F) = \pi(\rho)_F$, the base change of $\pi(\rho)$ to $\mathrm{GL}_N(\mathbb{A}_F)$. Let $L_{T'} \subseteq L$ be the field associated to T' and let ψ' be the character of $\mathrm{Gal}(LF/L_{T'}F)$ defined by restricting ψ. Then $\rho|_F \xrightarrow{\sim} \mathrm{Ind}_{LF}^{L_{T'}F}(\psi')$. Let $\tilde{\psi}'$ be the idele class character of $\mathbb{A}_{L_{T'}F}^*$ defined by ψ'. Then $\tilde{\psi}'$ is unramified and $\pi(\rho)_F = \mathrm{Ind}_F^{L_{T'}F}(\tilde{\psi}')$, where $\mathrm{Ind}_F^{L_{T'}F}$ denotes the automorphically induced representation of $\mathrm{GL}_N(\mathbb{A}_F)$ defined by $\tilde{\psi}'$ (cf. [AC], 3.6), using the fact that $\mathrm{Gal}(L_{T'F}/F)$ is a product of cyclic groups. It now follows at once from ([AC], 1.6.9) that $\pi(\rho|_F)$ is unramified at all finite places of F and $L_v(\pi(\rho|_F), s) = L_v(\rho|_F, s)$ for all finite places of F, since the corresponding identity holds for $L_w(\psi', s)$ and $L_w(\tilde{\psi}', s)$ at all finite places w of $L_{T'}F$. Hence $\pi(\rho|_F)$ is tempered at all finite places. Since $\pi(\rho|_F) = \pi(\rho)_F$ and $\pi(\rho)_{F,w}$ is tempered if and only if $\pi(\rho)_v$ is tempered for w dividing v ([AC], 1.6.4), $\pi(\rho)_v$ is tempered at all finite places. Let now η be a character of finite order of $\mathrm{Gal}(K^{ab}/K)$, with associated idele class character $\tilde{\eta}$. Suppose that $\eta_v = \chi_v$. Then $L_v(\pi(\rho) \otimes \tilde{\eta}, s) = L_v(\rho \otimes \eta, s)$ at almost all places, and since $(\pi(\rho)\otimes\tilde{\eta})_v$ is tempered everywhere, condition (b) of ([AC], 1.6.11) is satisfied relative to $L(\rho\otimes\eta,s)$ and $L(\pi(\rho)\otimes\tilde{\eta},s)$. We conclude

$$\prod_{v|p} L_v(\rho\otimes\eta,s)=\prod_{v|p} L_v(\pi(\rho)\otimes\tilde{\eta},s)$$

for all rational primes p and $p=\infty$. Fix a prime $p<\infty$ and choose v_0 lying over p. Let μ satisfy (a) $\tilde{\mu}_{v_0}\equiv 1$ and (b) $L_v(\pi(\rho)\otimes\tilde{\eta}\tilde{\mu},s)\equiv 1$ and $L_v(\rho\otimes\eta\mu,s)\equiv 1$ if $v|p$, $v\neq v_0$. (The existence of such $\tilde{\mu}$ follows from the Grunwald–Hasse–Wang theorem; it is sufficient to take $\tilde{\mu}_v$ highly ramified if $v|p$, $v\neq v_0$.) Then $L_{v_0}(\rho\otimes\eta,s)=L_{v_0}(\pi(\rho)\otimes\tilde{\eta},s)$ by the above identity. ∎

3.7 If $v|\infty$, we may argue as follows. Let F/K be a totally complex quadratic extension of K, if K is not totally complex, and let $F=K$, if K is totally complex. Then

$$\prod_{w|\infty} L_w((\rho\otimes\eta)|_F,s)=\Gamma_{\mathbb{C}}(s)^{Nd}$$

where $d=[F:\mathbb{Q}]/2$, and $\Gamma_{\mathbb{C}}(s)=2(2\pi)^{-s}\Gamma(s)$. Also,

$$\prod_{w|\infty} L_w((\pi(\rho)\otimes\tilde{\eta})_F,s)=\prod_{i=1}^{Nd}\Gamma_{\mathbb{C}}(s+\lambda_i)$$

with $\lambda_i\in\mathbb{C}$. Since these two products coincide, all λ_i are zero, and $L_w((\pi(\rho)\otimes\tilde{\eta})_F,s)=\Gamma_{\mathbb{C}}(s)^{Nd}$. Hence, if $v|\infty$ is a complex place of K, and $w|v$, then $L_v(\rho\otimes\eta,s)=L_w((\rho\otimes\eta)|_F,s)=L_w((\pi(\rho)\otimes\tilde{\eta})_F,s)=L_v(\pi(\rho)\otimes\tilde{\eta},s)$. If $v_0|\infty$ is real, then we can only conclude $L_{v_0}((\pi(\rho)\otimes\tilde{\eta})_F,s)=\Gamma_{\mathbb{R}}(s)^{a(v_0)}\Gamma_{\mathbb{R}}(s+1)^{b(v_0)}$ where $\Gamma_{\mathbb{R}}(s)=\pi^{-s/2}\Gamma(s/2)$. Let $L_{v_0}(\rho\otimes\eta,s)=\Gamma_{\mathbb{R}}(s)^{c(v_0)}\Gamma_{\mathbb{R}}(s+1)^{d(v_0)}$. Choose now F, $[F:K]=2$, so that v_0 splits in F but all other $v|\infty$ become complex in F. If w and $\bar{w}$ denote the places of F over v_0, then

$$\begin{aligned} L_w((\pi(\rho)\otimes\tilde{\eta})_F,s)L_{\bar{w}}((\pi(\rho)\otimes\tilde{\eta})_F,s)&=\Gamma_{\mathbb{R}}(s)^{2a(v_0)}\Gamma_{\mathbb{R}}(s+1)^{2b(v_0)},\\ L_w((\rho\otimes\eta)_F,s)L_{\bar{w}}((\rho\otimes\eta)_F,s)&=\Gamma_{\mathbb{R}}(s)^{2c(v_0)}\Gamma_{\mathbb{R}}(s+1)^{2d(v_0)}, \end{aligned}$$

but

$$\prod_{\substack{w'|\infty\\ w'\neq w,\bar{w}}} L_{w'}((\rho\otimes\eta)_F,s)=\Gamma_{\mathbb{C}}(s)^{N(d-1)}=\prod_{\substack{w'|\infty\\ w'\neq w,\bar{w}}} L_{w'}((\pi(\rho)\otimes\tilde{\eta})_F,s),$$

and hence we conclude $a(v_0)=c(v_0)$, $b(v_0)=d(v_0)$. Since $\eta_v=\chi_v=\tilde{\eta}_v$, the lemma is proved.

3.8 To prove the Proposition, note first that it follows at once that for each infinite place v, $\pi(\rho)_v \otimes \chi_v$ is the representation associated to $\rho|_{D_v} \otimes \chi_v$ by the Langlands correspondence. (This assertion makes sense since $D_v \tilde{\rightarrow} W_{K_v}^{ab}$ where W_{K_v} is the Weil group of K_v.) Since the class of $\rho_i|_{D_v}$ is independent of i in this case, (a) is true for infinite places. Further, if v is a finite unramified place, then the identity $L_v(\rho_i, s) = L_v(\rho_j \otimes \chi_{ij,v}, s)$ implies $L_v(\pi(\rho_i), s) = L_v(\pi(\rho_j) \otimes \chi_{ij,v}, s)$, and hence $\pi(\rho_i)_v \xrightarrow{\sim} \pi(\rho_j)_v \otimes \chi_{ij,v}$. Suppose that v is ramified for ρ. Since $\rho|_{D_v}$ is abelian, we can write

$$\rho|_{D_v} = \bigoplus_{i=1}^{t} m_i \chi_i$$

with distinct characters χ_i and multiplicitives m_i. Since $\pi(\rho)_v$ is tempered, we know, using Lemma (1.6.12) of [AC], that $\pi(\rho)_v = \mathrm{Ind}(\mathrm{GL}_n, P; \sigma_1, \dots, \sigma_r)$ for a parabolic $P = P(n_1, \dots, n_r)$, associated to partition $n = n_1 + \cdots + n_r$, and with cuspidal representations σ_j of the factors $\mathrm{GL}_{n_j}(\mathbb{A}_{K_v})$ of the Levi component of P. Further, $L(\pi(\rho)_v \otimes \chi, s) = \prod_{i=1}^{r} L(\sigma_i \otimes \chi, s)$, for each character χ of K_v^*. Since the local factor $(1 - q_v^{-s})^{-1}$ occurs with multiplicity m_i in $L((\rho|_{D_v}) \otimes \chi_i^{-1}, s)$, the same must be true of $L(\pi(\rho)_v \otimes \chi_i^{-1}, s)$. Since $L(\sigma_i \otimes \chi_i^{-1}, s) \equiv 1$ whenever $n_i > 1$, and since $L(\sigma_i \otimes \chi_i^{-1}, s) = (1 - q_v^{-s})^{-1}$ if and only if $\sigma_i = \chi_i$, we see that exactly m_i of the σ_j coincide with χ_i. Since the sum of the m_i is N, the set, counting multiplicities, of σ_i's must coincide with that of the χ_i's. To see (b), suppose $\pi(\rho_i) = \pi(\rho_j) \otimes \tilde{\chi}$ for some Hecke character $\tilde{\chi}$. Then $\tilde{\chi}$ has finite order since its values at almost all places reside in the roots of unity. Let χ be the associated Galois character. Then $L(\rho_i, s) = L(\rho_j \otimes \chi, s)$, and hence $\rho_i = \rho_j \otimes \chi$. As we have seen, this forces $i = j$ and thus (b) follows.

Finally if, $\rho = \rho_i$, then $\rho_i|_{D_v} = \rho_j|_{D_v} \otimes \chi_{ij,v}$, and hence $\pi_v(\rho_i) = \pi_v(\rho_j) \otimes \chi_{ij,v}$, since for any principal series representation defined by characters $\mu_1, \dots, \mu_N, \chi$, $\pi(\mu_1\chi, \dots, \mu_N\chi) = \pi(\mu_1, \dots, \mu_N) \otimes \chi$. This proves part (a) of the proposition.

3.9 It may be worth noting that the assumption that $\rho|_{D_v}$ is abelian can be weakened. Suppose $\rho = \rho' \otimes \rho_\Delta$, $\Delta \neq \{1\}$, and suppose $\rho'|_{D_v}$ is abelian. Then there exists a quadratic extension L_Δ of K such that $\rho_\Delta = \mathrm{Ind}_K^{L_\Delta}(\psi)$ with a character ψ: $\mathrm{Gal}(\bar{L}_\Delta/L_\Delta) \to \mathbb{C}^*$. Especially $\rho_\Delta|_{L_\Delta} \tilde{\rightarrow} \psi \oplus \psi \circ \tau =$ for τ the non-trivial automorphism of L_Δ/K. Put $\rho_j = \rho_\Delta \otimes \rho_j'$ with ρ_j': $H_4^b \times \prod_{i=1}^{m} H_{p_i} \to \mathrm{GL}_{N/2}(\mathbb{C})$. At each place where $\rho_\Delta|_{D_v}$ is abelian, we can conclude as before using twists that $\pi(\rho_i)_v = \pi(\rho_j)_v \otimes \chi_{ij,v}$ for a suitable $\chi_{ij,v}$. If $\rho|_{D_v}$ is non-abelian, we must note that $\pi(\rho_j)_{L_\Delta} \tilde{\rightarrow} \pi(\rho_j')_{L_\Delta} \otimes \tilde{\psi} \oplus \pi(\rho_j')_{L_\Delta} \otimes \tilde{\psi} \circ \tau$; this follows at once from the corresponding L-function identity, Theorem (4.2) of [AC], and the method employed above. Let w be the unique place of L_Δ which divides v. Then $\pi(\rho'_j)_{L_\Delta,w} = \pi(\varphi'_1,\ldots,\varphi'_{N/2})$ with characters φ'_j: $L^*_{\Delta,w} \to \mathbb{C}$, $\varphi'_j = \varphi_j \circ N_{L_\Delta,w/K_v}$, if $\pi(\rho'_j)_v = \pi(\varphi_1,\ldots,\varphi_{N/2})$. Hence

$$(\pi(\rho_j)|L_\Delta)_w = \pi(\varphi'_1\tilde{\psi}_w, \varphi'_1\tilde{\psi}_w \circ \tau, \ldots, \varphi'_{N/2}\tilde{\psi}_w, \varphi'_{N/2}\tilde{\psi}_w \circ \tau).$$

Let $I^{K_v}_{L_w}(\tilde{\psi}_w)$ be the unique supercuspidal representation of $\mathrm{GL}_2(K_v)$ which base changes to $\pi(\tilde{\psi}_w, \tilde{\psi}_w \circ \tau)$ on $\mathrm{GL}_2(L_{\Delta,w})$. Then, for the evident parabolic P, $\pi(\rho_j)_v \tilde{\to} \mathrm{Ind}(G, P,\ \mathrm{Ind}^{K_v}_{L_\Delta,w}(\tilde{\psi}_w) \otimes \varphi_1, \ldots,\ \mathrm{Ind}^{K_v}_{L_\Delta,w}(\tilde{\psi}_w) \otimes \varphi_{N/2})$ and hence (a) follows at v, since it holds for the $\pi(\rho'_j)_v$.

4. Proof of Theorem A

4.1 In 1–4 below we employ without proof the evident generalizations to SL_n of some results of Section 2 of [LL]. The proofs are, mutatis mutandis, the same as those in [LL]. Let v be a place of F, and let ψ: $F_v \to \mathbb{C}^*$ be a nontrivial continuous character. Let N be the unipotent radical of the standard Borel subgroup of $\mathrm{SL}_n(F_v)$. For $n \in N$, define

$$\psi_0(n) = \psi\left(\sum_{i=1}^{n-1} n_{i,i+1}\right)$$

if n has matrix entries n_{ij}. Let T_n be the group of diagonal matrices in $\mathrm{GL}_n(F_v)$ and define $\psi_0^t(n) = \psi_0(tnt^{-1})$ for $t \in T_n$. Every non-degenerate character ψ': $N(F) \to \mathbb{C}^*$ is of the form ψ_0^t for a suitable t. Recall that an irreducible admissible representation π_v of $\mathrm{SL}_n(F_v)$ is said to have a ψ_0^t-Whittaker model if

$$\mathrm{Hom}_{\mathrm{SL}_n(F_v)}(\pi_v, \mathrm{Ind}(\mathrm{SL}_n(F_v), N(F_v), \psi_0^t)) \neq 0$$

where

$$\begin{aligned}&\mathrm{Ind}\ (\mathrm{SL}_n(F_v), N(F_v), \psi_0^t)\\ &= \left\{f\colon \mathrm{SL}_n(F_v) \longrightarrow \mathbb{C} | f(ng) = \psi_0^t(n)f(g)\forall\ g \in \mathrm{SL}_n(F_v) \quad \text{and} \quad n \in N(F_v)\right\}.\end{aligned}$$

Every π_v which is a component of a cuspidal π has a ψ_0^t-Whittaker model for a suitable t.

4.2 Define $\mathcal{L}(\pi_v)$ as in the introduction. For $s \in T_n$, let π_v^s be the representation $g \mapsto \pi_v(s^{-1}gs)$. Then each irreducible constituent of $\mathcal{L}(\pi_v)$ is isomorphic to π_v^s for suitable $s \in T_n$. Further, if π_v is ψ_0^t-Whittaker, the π_v^s is ψ_0^{st}-Whittaker, as is shown by a trivial calculation. In fact, let $G(\pi_v) = \{s \in T_n | \ \pi_v^s \tilde{\rightarrow} \pi_v\}$. Then via det: $\mathrm{GL}_n(F_v) \rightarrow F_v^*$ and its splitting Δ: $F_v^* \rightarrow \mathrm{GL}_n(F_v)$, where $\Delta(f) = \mathrm{Diag}(f, 1, \ldots, 1)$, we see that $G(\pi_v)$ is naturally identified with a subgroup of F_v^* containing $(F_v^*)^n$, and the quotient $S(\pi_v) = F_v^*/G(\pi_v)$ is in bijection with the set of classes of constitutents of $\mathcal{L}(\pi_v)$, i.e. $s \in S(\pi_v)$ corresponds to $\pi_v^{\tilde{s}}$ where $\tilde{s}$ is any representive for s.

4.3 Let $\lambda_n = \mathrm{Diag}(-1, 1, -1, \ldots)$. Then if n is odd, or $4|n$, $\det(\lambda_n) = 1$. Thus, $\pi_v^{\lambda_n} \tilde{\rightarrow} \pi_v$. Note that if π_v has a ψ_0^t-Whittaker model, then its complex conjugate ${}^c\pi_v$ has a $\bar{\psi}_0^t$-Whittaker model. Since $\bar{\psi}_0^t = \psi_0^{t\lambda_n}$, we see that if $4|n$ or n is odd, ${}^c\pi_v$ also has a ψ_0^t-Whittaker model.

4.4 Let $\tilde{\pi}_v$ be an extension of π_v to $\mathrm{GL}_n(F_v)$. Let $X(\pi_v) = \{\chi\colon F_v^* \rightarrow \mathbb{C}^* | \tilde{\pi}_v \otimes \chi \tilde{\rightarrow} \tilde{\pi}_v\}$. Then $G(\pi_v) = \bigcap_{\chi \in X(\pi_v)} \mathrm{Ker}(\chi)$. Suppose that $\tilde{\pi}_v$ is unramified. Then $X(\pi_v)$ consists of unramified characters. In particular, $-1 \in G(\pi_v)$ since $-1 \in \mathrm{Ker}(\chi)$ for all $\chi \in X(\pi_v)$. Thus, if $\tilde{\pi}_v$ is unramified, then $\det(\lambda_n) \in G(\pi_v)$ even if $n = 2m$ with m odd. Hence if π_v has a ψ_0^t-Whittaker model, so does ${}^c\pi_v$, by the same argument as before.

4.5 Suppose now that $\pi_v = \pi(\rho_\alpha)_v$. Then ${}^c\pi(\rho_\alpha) = \pi({}^c\rho_\alpha) = \pi(\rho_\beta)$ for some $\beta \neq \alpha$, and hence ${}^c\pi(\rho_\alpha)_v \cong \pi(\rho_\alpha)_v \otimes \chi_v$ for a suitable χ_v. Thus $\mathcal{L}({}^c\pi_v) \cong \mathcal{L}(\pi_v)$, and if n is odd, $4|n$, or $\tilde{\pi}_v$ is unramified, ${}^c\pi_v \tilde{\rightarrow} \pi_v$ whenever $\tilde{\pi}_v = \pi(\rho_\alpha)_v$.

4.6 Return now to the global situation. Then ${}^c\pi(\rho_\alpha) \cong \pi({}^c\rho_\alpha) \cong \pi(\rho_\beta)$, for $\beta \neq \alpha$. But $\pi(\rho_\beta)$ is not a global twist of $\pi(\rho_\alpha)$. Thus, defining $L(\pi)$ as in the introduction, we have $L(\pi(\rho_\alpha)) \cap L(\pi(\rho_\beta)) = \{0\}$. Hence, if n is odd, $4|n$, or $\pi(\rho_\alpha)$ is unramified, Theorem A follows. Starting from a given F, we can, for a given ρ_α, find a compositum L of cyclic extensions of F such that $\rho_{\alpha|L}$ is unramified, but remains an irreducible representation. We can choose L in infinitely many disjoint ways. Thus Theorem A follows in general.

5. Proof of Theorem B

5.1 By [N], and the theorem of Shafarevich describing the maximal pro-p extension of a local field, we can find can find a Galois extension L of Q with

$\mathrm{Gal}(L/Q)\tilde{\to}H_p$ and such that $D_p = H_p$, where D_p is the decomposition subgroup for the place w of L which lies above p. Define a family ρ_α $(1 \leq \alpha \leq p-1)$ for this L and hence the representations $\tilde{\pi}(\rho_\alpha) \in A_0(\mathrm{GL}_p(\mathbb{A}_Q))$. As before, for a prime $\ell \neq p$, we have, for each pair α and β

$$\tilde{\pi}(\rho_\beta)_\ell \tilde{\to} \tilde{\pi}(\rho_\alpha)_\ell \otimes \chi_{\alpha\beta,\ell}.$$

Thus, if $\ell \neq p$, $\mathcal{L}_\ell(\pi(\rho_\alpha)) \cong \mathcal{L}_\ell(\pi(\rho_\beta))$. We must show that $\mathcal{L}_p$ $(\pi(\rho_\alpha))$ is not isomorphic to (and hence is disjoint from) $\mathcal{L}_p(\pi(\rho_\beta))$. Let ψ: $H_p \to \mathbb{C}^*$ be a character of order p. Then for all α, $\rho_\alpha \otimes \psi \tilde{\to} \rho_\alpha$, as one sees at once by considering conjugacy classes. Let $K \subsetneq L$ be the cyclic extension associated to $\mathrm{Ker}(\psi)$. Then $\rho_\alpha \tilde{\to} \mathrm{Ind}_Q^K(\chi_\alpha)$ and $\rho_\beta \tilde{\to} \mathrm{Ind}_Q^K(\chi_\beta)$ with characters χ_α, χ_β of $\mathrm{Gal}(L/K)$. Note that the center Z of H_p is contained in $\mathrm{Gal}(L/K)$, and $\chi_\alpha|_Z \neq \chi_\beta|_Z$ if $\alpha \neq \beta$. Identify χ_α and χ_β with idele class characters of $\mathbb{A}_K^*$. Then $\tilde{\pi}(\rho_\alpha) \cong \mathrm{Ind}_K^{\mathbb{Q}}(\chi_\alpha)$ and $\tilde{\pi}(\rho_\beta) \cong \mathrm{Ind}_K^{\mathbb{Q}}(\chi_\beta)$ where the $\mathrm{Ind}_K^{\mathbb{Q}}$ denotes again the automorphic induction of [AC]. Let v be the place of K lying over p. Then $[K_v\colon Q_p] = p$, $\chi_{\alpha,v} \circ \tau \neq \chi_{\alpha,v}$ and $\chi_{\beta,v} \circ \tau \neq \chi_\beta$ if τ generates $\mathrm{Gal}(K/Q)$. Hence $\tilde{\pi}(\rho_\alpha)_p$ and $\tilde{\pi}(\rho_\beta)_p$ are supercuspidal (by [AC]). Suppose now that $\tilde{\pi}_p(\rho_\alpha) \tilde{\to} \tilde{\pi}_p(\rho_\beta) \otimes \chi_{\alpha\beta,p}$. Then $\mathrm{Ind}_{K_v}^{Q_p}(\chi_{\beta,v}) \tilde{\to} \mathrm{Ind}_{K_v}^{\mathbb{Q}_p}(\chi_{\alpha,v}) \otimes \chi_{\alpha\beta} \tilde{\to} \mathrm{Ind}_{K_v}^{\mathbb{Q}_p}(\chi_{\alpha,v}(\chi_{\alpha\beta} \circ N_{K_v/\mathbb{Q}_p}))$. Hence, by [AC], there exists $a \in \mathbb{Z}$ such that

$$\chi_{\beta,v} \circ \tau^a = \chi_{\alpha,v}(\chi_{\alpha\beta,p} \circ N_{K_v/\mathbb{Q}_p}).$$

Since $\chi_{\alpha\beta,p}$ has finite order, we may identify it with a character of $\mathrm{Gal}(\bar{Q}_p/Q_p)^{ab}$. Then $\chi_{\alpha\beta,p}$ is a character of H_p and so $\chi_{\alpha\beta,p}$ is trivial on Z. However, $\chi_{\beta,v} \circ \tau^a|_Z = \chi_{\beta,v}|Z$; note that this makes sense since $Z \subset \mathrm{Gal}(L/K)$. Hence $\chi_{\alpha,v|Z} = \chi_{\beta,v}|_Z$ and so $\alpha = \beta$. Thus, if $\alpha \neq \beta$, $\mathcal{L}(\pi(\rho_\alpha)_p)$ is disjoint from $\mathcal{L}(\pi(\rho_\beta)_p)$. This proves the theorem.

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